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Comment on Spinor Anomalies

by

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Abstract

The structure of gauge anomalies for the fermion field coupled to general external fields is analyzed by a direct perturbative calculation in four dimensions. The anomalies associated with the nonabelian tensor fields are shown to cancel identically leaving only the pure gauge field anomalies.

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Chiral anomalies have played an important role in determining the structure of gauge field theories. These anomalies reflect the lack of local gauge symmetry of chiral fermions coupled to gauge fields. In this paper we will study the local gauge properties of spinor fields coupled to arbitrary external boson fields. We will focus on the anomalies associated with the coupling of antisymmetric tensor fields to the fermions.

The general structure of nonabelian chiral anomalies was first completely analyzed by Bardeen[1] through a direct calculation of the one fermion loop Feynman diagrams. In this calculation, explicit dependence of the anomalies on the external gauge, scalar, and pseudoscalar was determined. It was shown that all apparent anomalies associated with the scalar and pseudoscalar fields could be removed by appropriate counterterms and therefore could be considered as artifacts of the particular computational method used to evaluate the diagrams. Since all renormalizable field theories in four dimensions involve only the coupling of these fields to fermions, this previous analysis was sufficient to determine the anomaly structure of renormalizable field theory. However, it may be of interest to study the more general class of anomalies which include the antisymmetric tensor fields as such couplings may be generated as effective interactions of a more fundamental theory. Induced anomalous magnetic moments are a good example. The gauge consistency of the theory which includes these effective interactions is determined by absence of anomalies associated with the antisymmetric tensor fields.

The absence of anomalies involving abelian antisymmetric tensor fields was established by Clark and Love[2] using the BPHZ renormalization methods and by Bardeen and Gottlieb[3] using the original methods of Ref. 1. The extension of these results for the tensor anomalies to nonabelian case seems, at first, complex but may actually be obtained as a direct extension of the methods applied to the abelian case. In this paper we will present the results based on the original calculation of Bardeen[1] (known as -I- below), but the same results may be derived using the BPHZ methods.

In I, a spinor field is coupled to general vector, axial vector, scalar, and pseudoscalar external fields with nonderivative interactions and arbitrary internal symmetry. The Lagrangian density is given by

$$L = \bar{\psi}(z)[i\gamma\cdot\partial + \tilde{\Gamma}(z)]\psi(z) \qquad (1)$$

where the function $\tilde{\Gamma}(z)$ is a matrix in Dirac space and internal symmetry space. We make the following expansion in terms of fields of different spin

$$\tilde{\Gamma}(z) = -P_+(z) + \gamma_\mu V_+{}^\mu(z) + \sigma_{\mu\nu} T^{\mu\nu}(z) \tag{2}$$

with

$$\begin{aligned} P_\pm(z) &= M_0 + \Sigma(z) \pm i\gamma_5 \Pi(z) \\ V_\pm{}^\mu(z) &= V^\mu(z) \pm \gamma_5 A^\mu(z) \\ T_{\mu\nu}(z) &= T_{\mu\nu}(z). \end{aligned} \tag{3}$$

The vacuum functional for the general spinor loop was computed using a particular symmetric point-split regularization procedure and the Ward identities examined to confirm the existance of the spinor loop anomalies. In modern parlance, the gauge dependence of the regularized fermion determinant was computed and the consistent anomaly determined. Actually most of the gauge variance of the point-split vacuum functional could be removed by the proper choice of local counterterms and the conventional anomaly expression is obtained only after this freedom is exploited. The particular form of the counter terms needed is, of course, an artifact of the choice of regularization procedure and only the final result has an invariant meaning independent of this choice.

We will follow exactly the calculation in I to determine possible tensor anomalies. Although the original calculation of the unrenormalized vacuum functional and the related Ward identities was explicitly done for a restricted form of external fields, $\tilde{\Gamma}(z)$, it is actually valid for the general external fields shown in Eq.(2) which include the tensor fields. With a slight change in notation, we can adopt the results of these calculations.

The unrenormalized vacuum functional, $S_\epsilon(\tilde{\Gamma})$, is obtained from the expansion of the connected spinor loop diagrams,

$$\exp\{S_\epsilon(\tilde{\Gamma})\} = \langle 0 | T\{\exp[i\int dz(\overline{\psi}(z)\Gamma(z)\psi(z))_\epsilon]\} | 0\rangle \tag{4}$$

where $\Gamma(z) = \tilde{\Gamma}(z) + M_0$. M_0 is the mass used as the infrared regulator in the expansion of the loops.

The classical action is invariant under the following gauge transformation of $\tilde{\Gamma}(z)$,

$$\delta_\Lambda \tilde{\Gamma}(z) = \gamma\cdot\partial\Lambda_+(z) - \tilde{\Gamma}(z) i\Lambda_+(z) + i\Lambda_-(z)\tilde{\Gamma}(z) \tag{5}$$

or in terms of the component fields,

$$\begin{aligned}
\delta_\Lambda V_+{}^{\mu}(z) &= \partial^{\mu}\Lambda_+(z) - i[V_+{}^{\mu}(z), \Lambda_+(z)], \\
\delta_\Lambda P_+(z) &= -iP_+(z)\Lambda_+(z) + i\Lambda_-(z)P_+(z), \\
\delta_\Lambda T^{\mu\nu}(z) &= (1/2)i[\Lambda_+(z)+\Lambda_-(z), T^{\mu\nu}(z)] \\
&\quad + (1/2)\{\Lambda_+(z)-\Lambda_-(z), \tilde{T}^{\mu\nu}(z)\},
\end{aligned} \tag{6}$$

where the dual transform of $T^{\mu\nu}(z)$ is given by

$$\tilde{T}^{\mu\nu}(z) = (1/2)\, \varepsilon^{\mu\nu\sigma\tau}\, g_{\sigma\alpha}\, g_{\tau\beta}\, T^{\alpha\beta}(z), \tag{7}$$

and $\Lambda_\pm(z) = \Lambda(z)\pm\Lambda_5(z)$. The complicated transformation property of the tensor field is dictated by the chiral structure of the gauge transformation. *We note that the combinations $(T^{\mu\nu}(z)\pm i\tilde{T}^{\mu\nu}(z))$ have definite chirality.*

Under a general local gauge transformation, the connected vacuum functional transforms according to the result previously given in Eq.(I-29),

$$\delta_\Lambda S_\varepsilon(\tilde{\Gamma}) = D_\varepsilon(\Lambda_+, \tilde{\Gamma}). \tag{8}$$

$D_\varepsilon(\Lambda_+, \tilde{\Gamma})$ is the anomaly for this point-split definition of the vacuum functional and can be taken directly from Eq.(I-31). However this form of the vacuum functional may be simply renormalized by the counterterms *contained in $R_1(\tilde{\Gamma})$ as given in Eq.(I-37) and the first two terms in $R_2(\tilde{\Gamma})$ as* given in Eq.(I-39). These counterterms can be considered as functions of complete set of external fields, $\tilde{\Gamma}(z)$, including the tensor fields, and modify the form of the point-split anomaly through the relations,

$$\begin{aligned}
S_R(\tilde{\Gamma}) &= S_\varepsilon(\tilde{\Gamma}) - R_1(\tilde{\Gamma}) - \bar{R}_2(\tilde{\Gamma}) \\
\delta_\Lambda S_R(\tilde{\Gamma}) &\equiv D_R(\Lambda_+, \tilde{\Gamma}) = D_\varepsilon(\Lambda_+, \tilde{\Gamma}) - \delta_\Lambda R_1(\tilde{\Gamma}) - \delta_\Lambda \bar{R}_2(\tilde{\Gamma})
\end{aligned} \tag{9}$$

Using the results contained in I, we obtain the renormalized anomaly,

$$
\begin{aligned}
D_R(\Lambda_+,\tilde{\Gamma}) = (4\pi)^{-2}\, i\int dz\, \mathrm{tr}\{\, (1/36)\partial^\mu\Lambda_+(z)\, [\gamma_\mu\tilde{\Gamma}(z)\gamma_\nu\tilde{\Gamma}(z)\gamma^\nu\tilde{\Gamma}(z) \\
+ \gamma_\nu\tilde{\Gamma}(z)\gamma_\mu\tilde{\Gamma}(z)\gamma^\nu\tilde{\Gamma}(z) + \gamma_\nu\tilde{\Gamma}(z)\gamma^\nu\tilde{\Gamma}(z)\gamma_\mu\tilde{\Gamma}(z)] \\
+ (1/6)\, i\Lambda_+(z)\, [\gamma\cdot\partial\tilde{\Gamma}(z)\gamma\cdot\partial\tilde{\Gamma}(z) - \gamma^\nu\partial^\mu\tilde{\Gamma}(z)\gamma_\mu\partial_\nu\tilde{\Gamma}(z)]\}.
\end{aligned}
\qquad (10)
$$

This result for the renormalized anomaly is a simple exact form, but we have not fully exploited the freedom to add further counterterms. The result of Eq.(10) still contains contributions from the scalar, pseudoscalar, and tensor field components of $\tilde{\Gamma}(z)$ as well as the usual contributions of the vector and axial vector fields. In I, we showed that all dependence on the scalar and pseudoscalar fields could be removed by the proper choice of the additional counterterms as given by the remaining terms in Eq.(I-39). However that calculation ignored possible contributions of the tensor field components. We now do a systematic study of the contributions of all components of the external field.

The calculations are greatly simplified by the observation that the vacuum functional defined through Eq.(9) is invariant under global chiral symmetry transformations. The existance of anomalies reflect the lack of local chiral symmetry, but the anomaly derived in Eq.(10) vanishes for constant gauge parameter, $\Lambda_+(z) = \Lambda_+$, as the final terms are a surface integral. This global invariance is not necessarily shared by other forms of the anomaly as the possible local counterterms needed to transform from our form of the anomaly to the other forms may explicitly break the global symmetries. For example the vacuum functional computed using a Pauli-Villars or a BPHZ renormalization scheme is invariant under local vector gauge transformations but is not invariant under general global or local chiral transformations. Of course, the global chiral symmetries are preserved for the "left-right symmetric" form of the pure gauge anomaly derived in Eq.(I-41). A further simplification of using the form of the anomaly as given in Eq.(10) results from the fact that only triangle and box diagrams contribute to Eq.(10) while pentagon diagrams also contribute to the anomaly in the Pauli-Villars or BPHZ calculations.

Our subsequent computation requires the decomposition of the external field matrix in terms of its different spin components, P_+, V_+^μ, and $T^{\mu\nu}$ as given in Eq.(2). We calculate the terms in the anomaly of Eq.(10) which depend explicitly on the scalar, pseudoscalar, and tensor fields. For the quadratic terms, the PP and the TT terms vanish identically while a mixed PT term survives,

$$D_{2PT}(\Lambda_+,P_+,T) = -(2/3)\,(4\pi)^{-2}\, i \int dz\, \mathrm{tr}\, \{(\partial_\nu \Lambda_+(z)\partial_\mu P_-(z) + \partial_\mu P_-(z)\partial_\nu \Lambda_-(z))\,[T^{\mu\nu}(z) + i\gamma_5\, \tilde{T}^{\mu\nu}(z)]\}. \tag{11}$$

The chiral structure is obvious and an appropriate counterterm is easily determined to be,

$$R_3(V_+,P_+,T) = -(2/3)\,(4\pi)^{-2}\, i\int dz\, \mathrm{tr}\{(V_{+\nu}(z)\, \partial_\mu P_-(z) + \partial_\mu P_-(z)\, V_{-\nu}(z))\,[T^{\mu\nu}(z) + i\gamma_5 \tilde{T}^{\mu\nu}(z)]\} \tag{12}$$

The gauge variation of this counterterm produces terms which cancel the quadratic part of the mixed PT anomaly. However, the derivatives in the expression in Eq.(12) generate additional contributions to mixed VPT cubic anomaly which must be added to similar terms contained in the expansion of the anomaly in Eq.(10).

The different cubic components to the anomaly obtained after the addition of the counter term of Eq.(12) include contributions to the PPV, TTV, PTV amplitudes in addition to the pure gauge anomalies. All of these components can be removed by the appropriate choice of cubic counterterms. A systematic but straightforward analysis produces the following set of counterterms for these cubic anomalies,

$$\begin{aligned} R_4(V_+,P_+,T) = (4\pi)^{-2}\, i \int dz\, \mathrm{tr}\{ & (1/6)[V_{+\mu}(z)V_+{}^{\mu}(z)P_-(z)P_+(z)] \\ & + (1/12)[V_{+\mu}(z)P_-(z)V_-{}^{\mu}(z)P_+(z)] \\ & - (1/36)[V_+{}^{\mu}(z)T^{\alpha\beta}(z)V_-{}^{\nu}(z)T^{\sigma\tau}(z)](\gamma_\mu\sigma_{\alpha\beta}\gamma_\nu\sigma_{\sigma\tau}) \\ & - (1/18)[V_+{}^{\nu}(z)V_+{}^{\mu}(z)T^{\alpha\beta}(z)T^{\sigma\tau}(z)](\gamma_\nu\sigma_{\alpha\beta}\gamma_\mu\sigma_{\sigma\tau}) \\ & - [V_{+\nu}(z)P_-(z)V_{-\mu}(z)][T^{\mu\nu}(z) + i\gamma_5\tilde{T}^{\mu\nu}(z^0] \\ & - (1/3)[P_-(z)V_{-\mu}(z)V_{-\nu}(z)][T^{\mu\nu}(z) + i\gamma_5\tilde{T}^{\mu\nu}(z)] \\ & - (1/3)[V_{+\mu}(z)V_{+\nu}(z)P_-(z)][T^{\mu\nu}(z) + i\gamma_5\tilde{T}^{\mu\nu}(z)]\}. \end{aligned} \tag{13}$$

The contribution to the anomaly from the counterterm, R_4, comes only from the shift terms for the vector fields as all the commutator terms vanish due the exact global chiral symmetry. We also note that certain identities for the product of gamma matrices, $(\gamma_\mu\sigma_{\alpha\beta}\gamma_\nu\sigma_{\sigma\tau})$, are used to show anomaly cancelation. The addition of the counterterms, R_3 and R_4, cancels all anomalies involving the scalar, pseudoscalar and tensor fields. The PPVV terms in R_4 are the same as those given in Eq.(I-39). The minimal form of the anomaly, in left-right symmetric form, requires one further set of counterterms involving only the vector fields,

$$R_5(V_+) = (4\pi)^{-2}\, i \int dz\, \mathrm{tr}\{ -(1/36)[V_{+\mu}(z)V_+{}^{\mu}(z)V_{+\nu}(z)V_+{}^{\nu}(z) - (1/72)[V_{+\mu}(z)V_{+\nu}(z)V_+{}^{\mu}(z)V_+{}^{\nu}(z)] \}. \tag{14}$$

We summarize the results. The vacuum functional for the spinor loops was defined precisely as in I but for more general external fields. Actually much of the calculation in I may be directly used. The expression for the renormalized anomaly, $D_R(\Lambda_+,\tilde{\Gamma})$, in Eq.(10) is taken from the formulas derived in I. Including the contribution of tensor fields, the minimal anomaly is obtained by the addition of the specific counterterms, R_3, R_4, and R_5 given in Eqs.(12,13,14). The full vacuum functional is given by,

$$S(V_+,P_+,T) = S_R(\tilde{\Gamma}) - R_3(V_+,P_+,T) - R_4(V_+,P_+,T) - R_5(V_+,P_+,T). \tag{15}$$

This construction of the vacuum functional produces corresponding full anomaly,

$$\begin{aligned} D(\Lambda_+,V_+,P_+,T) &= D_R(\Lambda_+,\tilde{\Gamma}) - \delta_\Lambda R_3(\Lambda_+,V_+,P_+,T) - \delta_\Lambda R_4(\Lambda_+,V_+,P_+,T) \\ &\quad - \delta_\Lambda R_5(\Lambda_+,V_+,P_+,T) \\ &= (1/6)\, i \int dz\, i\epsilon_{\mu\nu\sigma\tau}\, \mathrm{tr}\{ \gamma_5[2i\Lambda_+(z)\partial^\mu V_+{}^\nu(z)\partial^\sigma V_+{}^\tau(z) \\ &\quad - \partial^\mu \Lambda_+(z)V_+{}^\nu(z)V_+{}^\sigma(z)V_+{}^\tau(z)]\} \end{aligned} \tag{16}$$

which is the result for the "left-right" symmetric anomaly given in Eq.(I-41). The inclusion of the external tensor fields does not change the form of the fundamental anomaly of the spinor field.

We have considered the most general, nonderivative coupling of fermions to external fields. We have shown, by explicit calculation, that the nonabelian anomaly is a property of the vector fields only and does not involve other spin external fields in four dimensions. That the vector fields provide the only obstruction to defining the spinor vacuum functional, or fermion determinant, can probably be seen more directly through the methods of differential geometry[4] which are beyond the scope of this paper.

The methods we have used to determine the structure of the nonabelian anomaly are by no means unique. We have mentioned the Pauli-Villars regularization method, and much of this calculation of the tensor anomalies has been checked using the BPHZ procedure for the spinor loops. Both of these methods suffer from the fact that the global chiral symmetry is not preserved by the calculational procedure which makes the analysis of the tensor field anomalies much more difficult than the analysis presented in this paper.

Our result establishes that the introduction of tensor field couplings preserves the local gauge symmetry of the spinor field theory. While this is largely a technical result, it does have some immediate implications. In considering the low energy effective field theory of a more fundamental dynamics, higher dimensional operators are generated. If the low energy field theory is a gauge theory, then the leading corrections to the theory may involve induced nonabelian anomalous magnetic moment couplings which have precisely the form of the tensor couplings we have analyzed. Our result establishes the gauge consistency of the low energy effective field theory for arbitrary magnetic couplings. Of course there may well be further applications in four and higher dimensions.

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